NUMBERS AND DIGITS

01 If $x = 1^2 + 2^2 + 3^2 + ... + 2021^2$ and
$y = 1 \times 3 + 2 \times 4 + 3 \times 5 + ... + 2020 \times 2022$, then find the value of $(x - y)$.

$x = 1^2 + 2^2 + 3^2 + ... + 2021^2$

$y = 1 \times 3 + 2 \times 4 + 3 \times 5 + ... + 2020 \times 2022$

$= 0 \times 2 + 1 \times 3 + 2 \times 4 + ... + 2020 \times 2022$

$= (1 - 1)(1 + 1) + (2 - 1)(2 + 1) + ... (2021 - 1)(2021 + 1)$

$= (1^2 - 1^2) + (2^2 - 1^2) + ... (2021^2 - 1^2)$

$\Rightarrow x - y = \{1^2 - (1^2 - 1)\} + \{2^2 - (2^2 - 1)\} + ... \{2021^2 - (2021^2 - 1)\}$

$\Rightarrow x - y = 1 + 1 + ... + 1$ (2021 times)

$\Rightarrow x - y = 2021$

02 How many zero's end the number $2^{300} \times 4^{400} \times 5^{600}$?

If a number is multiplied by 10, it ends in single 0

If a number is multiplied by 10^2, it ends in two 0's

Thus we can conclude that when a number is multiplied by 10^n, then the number contains 'n' number of zero's.

Using this concept, we solve the problem, consider $2^{300} \times 4^{400} \times 5^{600}$

We can write, 4^{400} as $(2^2)^{400} = 2^{800} = 2^{200} \times 2^{600}$

Replacing 4^{400} with $2^{200} \times 2^{600}$

i.e., $2^{300} \times 2^{200} \times 2^{600} \times 5^{600}$

$= 2^{500} \times (2 \times 5)^{600}$ [Since $a^m \times b^m = (ab)^m$] $= 2^{500} \times 10^{600}$

i.e., $2^{300} \times 4^{400} \times 5^{600} = 10^{600} \times 2^{500}$

According to the observation, the number will end in 600 zero's

03 If x and y are prime numbers which satisfy $x^2 - 2y^2 = 1$. Solve for x & y.

Given that $x^2 - 2y^2 = 1 \Rightarrow x^2 = 2y^2 + 1$

since $2y^2$ is even $\Rightarrow x^2$ must be odd

$\Rightarrow$ x must be an odd integer

Let $x = 2n + 1 \Rightarrow x^2 = (2n + 1)^2 = 4n^2 + 4n + 1 = 2y^2 + 1$

$\Rightarrow 4n(n + 1) = 2y^2 \Rightarrow y^2 = 2n(n + 1)$

$\Rightarrow y^2$ is even $\Rightarrow$ y is an even integer

$\Rightarrow$ y = 2 ($\because$ y is a prime number)

$\Rightarrow x^2 = 2(2)^2 + 1 \Rightarrow x^2 = 9$ or $x = 3$

$\therefore (x, y) = (3, 2)$

04 The operation $\star$ is defined for all non-zero numbers by $a \star b = \frac{a^2}{b}$. Determine $[(1 \star 2) \star 3] - [1 \star (2 \star 3)]$.

$$(1 \star 2) \star 3 = \frac{1^2}{2} \star 3 = \frac{1}{2} \star 3$$

$$\frac{\left(\frac{1}{2}\right)^2}{3} = \frac{\left(\frac{1}{4}\right)}{3} = \frac{1}{12} \quad \& \quad [1 \star (2 \star 3)] = 1 \star \left(\frac{2}{3}\right)^2 = 1$$

$$\star\, \frac{4}{3} = \frac{1^2}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

Therefore, $[(1 \star 2) \star 3] - [1 \star (2 \star 3)]$

$$= \frac{1}{12} - \frac{3}{4} = \frac{1}{12} - \frac{9}{12} = \frac{-8}{12} \text{ or } \frac{-2}{3}$$

05 What are three unequal positive rational numbers a, b and c for which $a + b + c = \frac{1}{a+b+c}$?

The only number > 0 equal to its reciprocal is 1,

so a + b + c = 1

The infinitely many correct answers each specify any 3 unequal rational whose sum is 1

One correct answer is $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

06 If x + y = 2021 and $\frac{1}{x} + \frac{1}{y} = 2021$, what is the value of xy ?

Since $2021 = \frac{1}{x} + \frac{1}{y}$

$\Rightarrow \frac{x+y}{xy} = \frac{2021}{1}$

$\Rightarrow \frac{2021}{xy} = \frac{2021}{1}$

$\Rightarrow xy = 1$

07 What is the sum of the digits in the number equal to the product ?

$11 \times 101 \times 10001 \times 100000001 \times 10000000000000001$

To specify the product, just rewrite it as

$(10 + 1)(10^2 + 1)(10^4 + 1)(10^8 + 1)(10^{16} + 1)$

$= 1 + 10 + 10^2 + 10^3 + \ldots.. 10^{31}$

$= 111111 \ldots.. 111$

Each of these 32 digits is a 1

so their sum is 32

(OR)

$$\frac{(10+1)(10^2+1)(10^4+1)(10^8+1)(10^{16}+1)}{1} = \frac{(10-1)}{(10-1)}$$

$$\frac{(10^2-1)(10^2+1)(10^4+1)(10^8+1)(10^{16}+1)}{10-1}$$

$$\frac{10^{32}-1}{10-1} = \frac{99\ldots..99}{9} = 11\ldots..11$$

The numerator $10^{32} - 1$ has 32 digits

$\therefore$ The digit sum is 32